Abstract: This paper considers the optimal control problem for linear hybrid automata. In particular, it is shown that the problem can be transformed into a constrained optimization problem whose constraints are a set of inequalities with quantifiers. Quantifier Elimination (QE) techniques are employed in order to derive quantifier free inequalities that are linear. The optimal cost is obtained using linear programming. The optimal switching times and optimal continuous control inputs are computed and used in order to derive the optimal hybrid controller. Our results are applied to an air traffic management example. Copyright © 2004 IFAC

Keywords: hybrid automata, quantifier elimination, time optimal control

1. INTRODUCTION

A mathematical model for hybrid systems is the hybrid automaton, which represents discrete components by finite-state machines and continuous components by real-numbered variables. An important subclass of hybrid systems is the so called rectangular hybrid automata (Preußig et al., 1998), where in each mode the continuous dynamics are given as constant differential inclusions. When linear continuous dynamics with both control and disturbance inputs are considered the so called linear hybrid automata (Xia et al., 2002) are derived.

The optimal control problem, particularly the time-optimal or weighted time-optimal problem is considered in this paper. We derive a hybrid control strategy that drives the system from an initial state into the target set with minimal cost. Assuming that there exists a trajectory starting from the initial state to the target set, the constraints that the trajectory has to satisfy are converted into a set of inequalities with quantifiers. A set of quantifier-free inequalities with the switching times as variables is obtained using quantifier elimination tool QEPCAD (Collins and Hong, 1991). As the quantifier free inequalities are linear, linear programming is used in order to obtain the optimal cost and the associated optimal switching times. Based on the optimal switching times, the optimal continuous control inputs are derived and an optimal controller is constructed.

This paper is organized as follows. In section 2, we review the basic definitions and concepts related to linear hybrid automata. In section 3 we employ quantifier elimination for the optimal control of rectangular hybrid automata and design the corresponding optimal controller. In section 4, the optimal control problem for linear hybrid automata is solved. The air traffic management application is illustrated in section 5. Section 6 contains conclusions.

2. MODEL AND PROBLEM STATEMENT

2.1 Hybrid Automata

Hybrid automata are generalized finite-state machines where discrete transitions transfer the sys-
tem between a finite number of modes \( Q \). We use \( A \) to denote a hybrid automaton. In each discrete mode \( q \), the system evolves continuously according to a set of differential equations:

\[
\dot{x}_q = f_q(x(t), u(t), d(t))
\]  

where \( x \in X \subseteq \mathbb{R}^n \), \( u(t) \in U \) is the continuous control input and \( d(t) \in D \) is disturbance input. It is assumed that for every \( q \in Q \), \( f_q \) is globally Lipschitz to ensure the existence and uniqueness of solutions of the differential equation. Furthermore for each mode \( q \) there may be a set of state invariant constraints \( \text{Inv} : Q \to 2^X \) that have to be fulfilled. \( \Sigma \) is the set of discrete events and the discrete transition set is \( E \subseteq Q \times \Sigma \times Q \). We use guard \( G : (Q \times \Sigma \times Q) \to 2^X \) to express the switching conditions that when \( x \in G_{qq}(\sigma) \) the system can be switched from \( q \) to \( q' \) by a discrete transition.

Depending on the characteristics of dynamics \( \dot{x}_q = f_q(x, u, d) \) the following subclasses of hybrid automata can be defined:

- **Linear Hybrid Automata** (Xia et al., 2002): \( \dot{x}_q(t) = u(t) + d(t) \) where \( u(t) \in U_q \) and \( d(t) \in D_q \). \( U_q = \{ u \in \mathbb{R}^n \mid \text{h}_{U_q} \subseteq C_{U_q} \subseteq \mathbb{R}^n \} \) and \( D_q = \{ d \in \mathbb{R}^n \mid \text{h}_{D_q} \subseteq C_{D_q} \subseteq \mathbb{R}^n \} \). The continuous dynamics have both control and disturbance input.

- **Rectangular Hybrid Automata** (Preuß et al., 1998): \( \dot{x}_q(t) = u(t) \) where \( u \in [L_q, U_q] \). The continuous dynamics are rectangular differential inclusions.

- **Integrator Hybrid Automata** (Xu and Antsaklis, 2003): \( \dot{x}_q(t) = k_q \). The continuous variables evolve at a fixed speed.

- **Time Automata** (Alur et al., 2001): \( \dot{x}_q(t) = 1 \). The continuous variables \( x \) act as clocks.

For these types of hybrid automata, we assume that all the invariant sets, guard sets and target set are convex polyhedral sets. It is assumed that the continuous states are the same before and after a discrete transition i.e. \( q(t^-) = q(t^+) \), with \( x(t^-) = x(t^+) \). Also, we assume that all the discrete events are controllable.

We use \( \xi \) to denote a hybrid trajectory. The set of hybrid trajectories starting from \((q, x)\) is denoted by \( L(A, (q, x)) \) while \( L(A) \) denotes the set of all hybrid trajectories generated by the automaton \( A \).

The running cost introduced above involves the following two cost functions:

- \( J_d : E \to Q \) discrete transition cost function.
- \( J_c : Q \to Q \) continuous transition cost function.

Basiclly we assign costs to both discrete and continuous transitions. Note that there is no penalty on the control input and on the continuous state.

From definition (3), it is clear that if there is no trajectory from \((q_0, x_0)\) to \( F \) without violating the invariant constraints, then the optimal cost \( \min_{\xi \in L(A, (q_0, x_0))} J(\xi) = +\infty \). In order to guarantee that the optimal cost is finite, a backward reachability analysis (Xia et al., 2002) is performed starting from the target set \( F \). If the target is reachable from \((q_0, x_0)\), then the optimal cost is finite. Due to the fact that all discrete transitions are controllable, the procedure to find an optimal problem statement

The optimal control problem we considered in this paper is as follows: Given an initial state \((q_0, x_0)\) of a hybrid automaton \( A \) and a cost function \( J \) of a hybrid trajectory, find an optimal hybrid controller such that the controlled trajectory starting from \((q_0, x_0)\) reaches the target set \( F \) with a minimal cost. The solvability of the above problem depends on the definition of the cost function and the characteristics of the given automaton \( A \).

Given a trajectory \( \xi \in L(A, (q_0, x_0)) \) and the target set \( F \subseteq X \times Q \), the time that a trajectory \( \xi \) starting from \((q_0, x_0)\) needs to reach \( F \) is:

\[
T_f(\xi, F, (q_0, x_0)) = \begin{cases} 
  t_f & \text{if } \xi(0) = (q_0, x_0), \xi(t_f) \in F \\
  +\infty & \text{otherwise}
\end{cases}
\]  

if \( \forall t \in [0, t_f], \xi(t) \in INV \land \xi(t) \notin F \).

Given a trajectory \( \xi \in L(A, (q_0, x_0)) \), since we are only interested in the running cost incurred when the system evolves from \((q_0, x_0)\) to \( F \), we compute the cost along \( \xi \) during the time interval \([0, T_f(\xi, F, (q_0, x_0))]\). Let \( c_j(\xi) \) denote the \( j \)-th discrete transition and \( n(\xi) \) denote the total number of discrete transitions of \( \xi \) during the time interval \([0, T_f(\xi, F, (q_0, x_0))]\). Then, the cost along \( \xi \) is defined as:

\[
J(\xi) = \begin{cases} 
  \sum_{j=1}^{n(\xi)} J_d(c_j(\xi)) + \int_{0}^{t_f} J_c(q(t)) dt & \text{if } T_f(\xi, F, (q_0, x_0)) \text{ is finite} \\
  +\infty & \text{otherwise}
\end{cases}
\]  

(3)
The graph of $i$

Proof: If $\pi_i = \emptyset$ for $i \in \mathbb{N}$, this means that the graph of $A$ has a finite number of discrete paths, then it is obvious that algorithm 1 will terminate with finite iterations. On the other hand, as it is feasible to reach the target from $(q_0, x_0)$, then $J^* (\xi)$ is finite. Since $J_{\text{min}}$ is monotonously decreasing and $J^*_d$ is monotonously increasing during each iteration, it can be induced that there exists an integer $i$ such that $J^{*i} = J^*_d + J^*_c$ is optimal.

From algorithm 1, the attainability of the optimal solution for the optimal control problem depends on one’s ability to derive $J^*_c$. It is a continuous cost minimization problem along the fixed path $\pi$. This problem is considered in the rest of the paper.

Remark 3. Notice that if $J_c(q_j^*) = 1$ for all $j = 0, \ldots, l(\pi)$ along the path $\pi$, the optimal control problem is transformed into a time-optimal control problem as follows: $t_{\text{min}} = J^*_c = \min t_j \geq 0 \sum_{j=0}^{l(\pi)} t_j$, and $t_{\text{min}}$ is the minimal time that the hybrid automaton needs to reach the target set $F$ starting from $(q_0, x_0)$ along the path $\pi$. Otherwise the problem can be considered as a weighted time-optimal control problem. The overall cost is a balance between discrete transitions cost and time cost.

Let $G_{q_{i-1}, q_i} = \{x|C_i^g x \leq h_i^g\}$ denote the transition guard from $q_{i-1}^g$ to $q_i^g$, with $i = 1, \ldots, l(\pi)$, where $C_i^g \in \mathbb{R}^{n_u \times n}$, $h_i^g \in \mathbb{R}^{n_u}$. Also, let $X_F = \{x|C_F x \leq h_F\}$ be the continuous part of the target set, with $C_F \in \mathbb{R}^{n_x \times n}$, $h_F \in \mathbb{R}^{n_x}$. For simplicity, we omit the symbol “$\pi$”. Consider a discrete path $\pi$ with $l(\pi)$ transitions. Our objective is to compute the minimal cost $J^*_c$. The time-optimal problem for integrator hybrid automata has been studied in (Xu and Antsaklis, 2003). Next, the optimal control for rectangular hybrid automata and linear hybrid automata is considered.

3. OPTIMAL SOLUTION FOR RECTANGULAR HYBRID AUTOMATA

This section deals with the optimal control problem for rectangular hybrid automata. Before proceeding with the main theorem, we discuss the relationship between the open and closed loop control. In each discrete state $q$, there are two types of controllers that generate the control input, namely the open loop controllers and the closed loop (feedback) controllers. Let $u_q \in [\underline{U}_q, \overline{U}_q]$ denote the control input generated by an open loop controller and let $u_q(t) \in [\underline{U}_q, \overline{U}_q]$ denote the control input (time-variant) generated by a closed loop controller. The relationship between these two controllers follows the First Mean Value Theorem for Integrals, see (Pang and Spathopoulos, 2004).

It follows that along a path $\pi$, for any trajectory $\xi$, a feedback control input $u(t)$, an alternative trajectory $\xi'$ can always be found with open loop
control input $u$, such that $J(\xi) = J(\xi')$ and 
$\xi(\sum_{j=0}^{l-1} t_j) = \xi'(\sum_{j=0}^{l-1} t_j)$ for $j = 0, \ldots, l(\pi)$ where 
t$_j$ is the time the system spends in each location 
of the path $\pi$. Therefore, both type of controllers 
yield the same optimal cost. The advantage in the 
above consideration is that each open loop control 
$u_{qi}$ can be treated as an existential ($\exists$) quantifier. 
Thus, quantifier elimination can be employed in 
order to get a set of linear inequalities for the 
variable $t$.

The objective of the real quantifier elimination 
is to eliminate "unwanted" variables from an 
algebraic description. The "unwanted" variables 
may represent unknown real quantities. Quantifiers 
give expressive power but do not enlarge the 
class of sets defined by quantifier-free formulas. 
This implies that given a formula including quantifiers 
$\varphi(x_1, \ldots, x_n) \equiv Q_1u_1 \cdots Q_mu_m \psi(x_1, \ldots, x_n, u_1, \ldots, u_m)$ 
where $Q_i \in \{\exists, \forall\}$, there is always a logically 
equivalent quantifier free formula $\phi(x_1, \ldots, x_n)$ 
in the domain of the real numbers. A procedure 
computing such $\phi$ from $\varphi$ is called real quantifier 
elimination.

Given a rectangular hybrid automaton $A$, let the 
initial state be $x_0 = (x_{01}, \ldots, x_{0n})$, the 
dynamics of a discrete state $q_j \in \pi$ be $\dot{x}_j = u_j = 
[u_{j1}, u_{j2}, \ldots, u_{jn}]^T$ for all $j = 0, \ldots, l(\pi)$, with 
control input $u_j \in [U_j, \bar{U}_j]$, $U_j, \bar{U}_j \in \mathbb{R}^n$.

**Theorem 4.** The optimal continuous cost along 
a path $\pi$ for a rectangular automaton $A$ with 
parameters defined above, can be computed by 
$$ J^\pi_c = \min_{\phi(t)} \int_{t^*} \phi(t) $$ 

The quantifier free formula $\phi(t)$ that is linear in 
t is computed from $\varphi(t)$ by quantifier elimination 
real domain where: 
$$ \varphi(t) \equiv \exists u_{i1}, \ldots, \exists u_{in}, \exists u_{21}, \ldots, \exists u_{2n}, \ldots \exists u_{l(\pi)1}, \ldots, \exists u_{l(\pi)n}(\psi_1(u, t) \land \psi_2(u, t) \land \psi_3(t) \land \psi_4(u)) $$ 
and:

$$ \psi_1(u, t) = \bigwedge_{l(\pi)} \sum_{i=1}^{l(\pi)} C_i(x_{01}, \ldots, x_{0n})^T + 
\sum_{j=0}^{l(\pi)} [u_{j1}, \ldots, u_{jn}]^T t_j \leq h_i(\text{guard condition}) $$ 

$$ \psi_2(u, t) = \bigwedge_{l(\pi)} \sum_{j=0}^{l(\pi)} [u_{j1}, \ldots, u_{jn}]^T t_j \leq h_F(\text{target condition}) $$ 

$$ \psi_3(t) = \bigwedge_{j=1}^{l(\pi)} t_j \geq 0(\text{nonnegative time condition}) $$ 

$$ \psi_4(u) = \bigwedge_{j=1}^{l(\pi)} \bigwedge_{r=1}^{u_j} U_{jr} \leq u_{jr} \leq \bar{U}_{jr}(\text{input restrictions}) $$

**Proof:** See (Pang and Spathopoulos, 2004). \[ \square \]

It should be noted that since the two variables 
t and $u$ are nonlinear in the formulas $\psi_1 \sim \psi_4$, 
it is impossible to solve the LP problem. Nevertheless, 
by considering the continuous input $u$ as 
estential ($\exists$) quantifier, we get an equivalent 
formula $\phi$ by eliminating the $u$ quantifier. 
According to the quantifier elimination procedure, 
the quantifier free formula $\phi$ is formulated as linear 
inequalities whose closure is a convex set.

As mentioned early on, in order to guarantee the 
optimal performance, an optimal controller has to be 
 imposed on the system. As the LP (5) converges to 
an optimal solution, we get the optimal control 
switching times $t^*$. Then the continuous 
optimal control law $U^*$ can be obtained using the 
following two-step procedure:

- **Step1:** substitute the optimal time $t^*$ derived 
  from LP into the formulas $e_1, e_2, e_4$. 
- **Step2:** simplify these formulas to eliminate 
  redundant inequalities.

With the optimal times $t_1^*, \ldots, t_{l(\pi)}^*$ obtained from 
the LP problem 4, the optimal controller is:

$$ C^*(q(t), x(t), t) = \begin{cases} 
(\epsilon, u_0^*) & \text{if } 0 \leq t < t_0^* \\
\land q(t) = q_0 \land u_0^* \in U_0^* \\
(\sigma_{j,j+1}, D) & \text{if } t = \sum_{i=0}^{j} t_i^* \land q(t^-) = q_j \\
\land (q_j, \sigma_{j+1,j+1}) \in E \\
\text{for } j = 0, \ldots, l(\pi) - 1 \\
(\epsilon, u_j^*) & \text{if } \sum_{i=0}^{j} t_i^* \leq t < \sum_{i=0}^{j+1} t_i^* \\
\land q(t) = q_j \land u_j^* \in U_j^* \\
\text{for } j = 1, \ldots, l(\pi) 
\end{cases} $$

It should be noted that the set $U_j^*$, obtained 
from above two-step procedure, is the optimal 
continuous input set for the discrete state $q_j^*$. 
The clock is set to 0 at the initial state $(q_0, x_0)$. 
During the time $t \in [\sum_{j=0}^{i-1} t_j^*, \sum_{i=0}^{j} t_i^*]$ there is 
no discrete input and the continuous input is 
$u_j^* \in U_j^*$. At the optimal time $t = \sum_{i=0}^{j} t_i^*$ the 
continuous optimal controller generates a suitable 
discrete event to switch the system to the next 
discrete state $q_{j+1}$ along the path $\pi$. At the same 
time the continuous optimal controller chooses a 
continuous input from $U_{j+1}^*$ as soon as the system 
has been switched to the discrete state $q_{j+1}$.

4. OPTIMAL SOLUTION FOR LINEAR 
HYBRID AUTOMATA

If we consider linear hybrid automata with both 
control and disturbance inputs the problem becomes 
more difficult. The optimal control problem 
can not be solved using quantifier elimination 
when a feedback control is required. However, we
can still get an open loop controller in the sense that the control input \( u \) is fixed at each discrete state \( q \).

Due to the presence of disturbances, the open loop control is optimal in the worst case sense. The dynamics in a discrete state \( q_i \in \pi \) become

\[
\dot{x}_j(t) = u_j + d_j(t) \quad \text{for all } j = 0, \ldots, l(\pi),
\]

with control input \( u_j \in U_j, U_j = \{u \in \mathbb{R}^{n \times n}, h_{U_j}, C_{U_j} \in \mathbb{R}^{n \times n} \} \) and disturbance \( d_j(t) \in D_j \), \( D_j = \{d \in \mathbb{R}^{n \times n}, C_{D_j} \in \mathbb{R}^{n \times n}, h_{D_j} \in \mathbb{R} \} \).

Theorem 5. The worst case optimal continuous cost along a path \( \pi \) from an initial state \( x_0 \) for linear hybrid automata with both control and disturbance input can be computed as

\[
J^*_\pi = \min_{s.t. \phi(t)} \lambda^T t
\]

The quantifier free formula \( \phi(t) \) is computed from \( \varphi(t) \) by quantifier elimination in real domain, where

\[
\varphi(t) \equiv \exists u_{1,1}, \ldots, \exists u_{1,n}, \exists u_{2,1}, \ldots, \exists u_{2,n}, \ldots, \exists u_{l(\pi),1}, \ldots, \exists u_{l(\pi),n}(\psi_1(u, t) \land \psi_2(u, t) \land \psi_3(t) \land \psi_4(u))
\]

and:

\[
\psi_1(u, t) = \bigwedge_{l(\pi)} \bigwedge_{j=0}^{l(\pi)} \bigwedge_{m=1}^{p_j} \bigwedge_{m=1}^{p_j} (C_i(x_0 + \sum_{j=0}^{l(\pi)} (u_j + d_j) t_j) \leq h_i)
\]

\[
\psi_2(u, t) = \bigwedge_{m=1}^{p_j} \bigwedge_{m=1}^{p_j} C_F(x_0 + \sum_{j=0}^{l(\pi)} (u_j + d_j) t_j) \leq h_F
\]

\[
\psi_3(t) = \bigwedge_{j=1}^{l(\pi)} t_j \geq 0
\]

\[
\psi_4(u) = \bigwedge_{j=1}^{l(\pi)} C_{U_j} u_j \leq h_{U_j}
\]

Proof: See (Pang and Spathopoulos, 2004).

The optimal continuous control law \( U^* \) for linear hybrid automata can also be designed following the two-step procedure presented above. The optimal controller \( C^* \) is the same with the one derived for rectangular hybrid automata as in (6).

5. APPLICATION OF AIR TRAFFIC MANAGEMENT

In this section, the results of optimal control are applied on an Air Traffic Management System (ATMS) (Tomlin, 1996), (Pang and Spathopoulos, 2004). This two aircraft joining and collision avoidance scheme can be seen as an optimal control problem. The scenario is a common one and illustrated in figure 5. Aircraft 2 is joining the path of Aircraft 1, in approach to the airport, and it is up to the controller to ensure that a minimal separation distance between the two aircraft is maintained and also minimize the cost during this procedure. Aircraft 1 has one mode of operation, called Cruise: it follows a straight path along the x-axis and has known bounds on its speed. Aircraft 2 is in three possible position modes: Approach, Turn and Join. Within the Turn and Join modes, Aircraft 2 can be in one of three possible speed modes, the controller effects both the transitions and the speed of the modes. The control scheme is as follows: if the x coordinate of Aircraft 2 is either greater than that of Aircraft 1 or more than 30 units behind that of Aircraft 1, then the Aircraft 2 should be in a fast mode. If the x coordinate of Aircraft 2 is less than 30 units behind that of Aircraft 1, then the Aircraft 2 should be in a slow mode. There are two fast modes and one slow mode: Fast1 is the mode in which Aircraft 2 is more than 30 units behind Aircraft 1, Fast2 is the mode in which Aircraft 2 is ahead of Aircraft 1.

The speed of Aircraft 1 is \( \dot{x} = u_1 \), where \( u_1 \in [3, 4] \) in any position mode. For simplicity, we restrict the transitions between the Fast and slow modes if they are in the same position mode. Then the aim of this air traffic management is that Aircraft 2 join the path of Aircraft 1 safely with \( x_1 \geq 50 \land x_2 \geq 50 \). The safety requirement is that the distance between the two aircraft has to be kept over 10. That is roughly illustrated by the forbidden region \( Bad = \{(x_1-x_2) \leq 10 \land y^2 \leq 10\} \).

It should be noted that the problem requires the composing the motions of both Aircraft 1 and Aircraft 2. The composed automaton has three discrete position modes of Aircraft 2. Its continuous states are in 3 dimensions: \( (x_1, x_2, y_2) \).

We assume that the two aircraft communicate very well, and the Aircraft 1 can also change its speed when Aircraft 2 changes its position modes. Hence, by composing the two systems we get the target set \( F = (Join, F_1 \cup F_2) \) for the composed hybrid system where the discrete state of Aircraft 2 is Join and the continuous sets are either \( F_1 = \{x_1-x_2 \geq 10 \land x_2 \geq 50 \} \) or \( F_2 = \{x_2-x_1 \geq 10 \land x_1 \geq 50 \} \).
The optimal control for this air traffic management is cast as follows: the two aircraft reach the target set $F$ without entering the forbidden region $Bad$ at minimal cost. Here, we associate each transition with the discrete cost $J_d = 20$. The continuous costs in $Approach$ and $Join$ position modes are $J_c(\text{Approach}) = J_c(\text{Join}) = 2$, and the continuous cost in the $Turn$ position mode is $J_c(\text{Turn}) = 3$. Given an initial state in $Approach$ mode: $x(0) = -30, x_2(0) = 0, y_2(0) = -100$, there are only two discrete paths acceptable (the others generate blocking or lead the system to the forbidden region), namely $\pi_1 = \{\text{Appf \ Turn f} \ J oin\}$ and $\pi_2 = \{\text{Appf \ Turn f2} \ J oin f2\}$. The details of these paths are shown in table 1. It follows that the optimal cost is $J^{\ast}_2 = 90$ with the optimal schedule $t^* \pi_2$ as shown in table 1. The optimal continuous inputs of Aircraft 1 and Aircraft 2 along the path $\pi_2$ are illustrated in figure 2 and the optimal trajectories $\xi^* \pi_2$ in 3D are shown in figure 2.

### Table 1. Final results of two paths.

<table>
<thead>
<tr>
<th>Path</th>
<th>$\pi_1$</th>
<th>$\pi_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_0$</td>
<td>$x_2 = 0$</td>
<td>$x_2 = 0$</td>
</tr>
<tr>
<td></td>
<td>$y_2 \in [-30, 30]$</td>
<td>$y_2 \in [-30, 30]$</td>
</tr>
<tr>
<td></td>
<td>$x_1 \in [-40, 40]$</td>
<td>$x_1 \in [-40, 40]$</td>
</tr>
<tr>
<td>$q_1$</td>
<td>$\text{Turn f}$</td>
<td>$\text{Turn f2}$</td>
</tr>
<tr>
<td></td>
<td>$0 \leq x_2 \leq 30$</td>
<td>$0 \leq x_2 \leq 30$</td>
</tr>
<tr>
<td></td>
<td>$-30 \leq y_2 \leq 30$</td>
<td>$-30 \leq y_2 \leq 0$</td>
</tr>
<tr>
<td></td>
<td>$x_2 \leq x_1 \leq 30 + x_2$</td>
<td>$x_2 \geq 30$</td>
</tr>
<tr>
<td></td>
<td>$\exists x \in [3, 4]$</td>
<td>$\exists x \in [3, 4]$</td>
</tr>
<tr>
<td>$q_2$</td>
<td>$\text{Join}$</td>
<td>$\text{Join f2}$</td>
</tr>
<tr>
<td></td>
<td>$x_2 \geq 30$</td>
<td>$x_2 \geq 30$</td>
</tr>
<tr>
<td></td>
<td>$x_2 \leq x_1 \leq 30 + x_2$</td>
<td>$x_2 \geq x_1$</td>
</tr>
<tr>
<td></td>
<td>$\exists x \in [4, 5]$</td>
<td>$\exists x \in [5, 6]$</td>
</tr>
<tr>
<td></td>
<td>$y_2 = x_2$</td>
<td>$y_2 = 0$</td>
</tr>
<tr>
<td></td>
<td>$\exists x \in [3, 4]$</td>
<td>$\exists x \in [3, 4]$</td>
</tr>
<tr>
<td>$G_{\pi_1 \pi_2}$</td>
<td>$x_2 \leq x_1 \leq 30 + x_2$</td>
<td>$x_2 \geq x_1$</td>
</tr>
<tr>
<td></td>
<td>$y_2 \leq 0$</td>
<td>$y_2 = 0$</td>
</tr>
<tr>
<td>$G_{\pi_1 \pi_2}$</td>
<td>$x_2 \leq 0$</td>
<td>$x_2 = 0$</td>
</tr>
<tr>
<td></td>
<td>$10 \leq x_2 - x_1 \leq 30$</td>
<td>$x_2 - x_1 \geq 10$</td>
</tr>
<tr>
<td>Target</td>
<td>${\text{Join f, f1}}$</td>
<td>${\text{Join f2}, f2}$</td>
</tr>
<tr>
<td></td>
<td>$[20 + 20, 40]$</td>
<td>$[20 + 20, 40]$</td>
</tr>
</tbody>
</table>

### Table 2. Optimal controller for path $\pi_2$

<table>
<thead>
<tr>
<th>Mode</th>
<th>Controller</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Join f}$</td>
<td>$x_1 = u_{10} = 3, y_2 = u_{20} = 7$</td>
</tr>
<tr>
<td>$\text{Join f2}$</td>
<td>$x_1 = u_{11} = 4, x_2 = y_2 = u_{21} = 6$</td>
</tr>
<tr>
<td>$\text{Turn f}$</td>
<td>$x_1 = u_{11} = 4, x_2 = y_2 = u_{21} = 6$</td>
</tr>
<tr>
<td>$\text{Turn f2}$</td>
<td>$x_1 = u_{11} = 4, x_2 = y_2 = u_{21} = 6$</td>
</tr>
</tbody>
</table>

Fig. 2. Optimal trajectories in 3D.

have relative simple dynamics. Optimal control for the decidable classes of linear hybrid systems stated in (Lafferriere et al., 2001) should be further investigated.

7. REFERENCES


