Non-linear electrodynamics: A classical view of the quantum vacuum

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The linear vacuum

In the linear vacuum an electromagnetic field, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), will propagate according to Maxwellian electrodynamics. From this field there are two Lorentz invariant quantities which can be obtained, as:

\[
X = -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (E^2 - B^2)
\]

\[
Y = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = (E \cdot B)
\]

where \( F^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}/2 \) is the dual electromagnetic field. The field equations can be obtained from the action,

\[
S = \int d^4x \sqrt{-g} \mathcal{L}(F, \partial F),
\]

and are given by:

\[
\partial_\mu F^{\mu\nu} = 0 \quad \text{and} \quad \partial_\mu H^{\mu\nu} = 0.
\]

These equations have been extensively studied – Euler-Heisenberg [2] and Born-Infeld [3].

Non-linear theories

Non-linear vacuum electrodynamics theories aim to describe high intensity processes which are not captured by the theory of Maxwellian electrodynamics. They are defined by a generalization of the action,

\[
S = \int d^4x \sqrt{-g} \mathcal{L}(X, Y),
\]

where \( \mathcal{L}(X, Y) \) is an arbitrary function of the Lorentz invariants. Varying the action with respect to the gauge field \( A_\mu \) gives a modification to Maxwell’s equations, which allow for the non-linear interaction of electromagnetic fields,

\[
\partial_\mu H^{\mu\nu} = 0.
\]

The excitation tensor characterises the non-linear interaction between electromagnetic fields, which causes waves moving through a strong background to behave in an analogous way to light interacting with a medium (see e.g. [1] for review).

In the literature two particular non-linear theories have been extensively studied – Euler-Heisenberg [2] and Born-Infeld [3].

Conformal invariance

Quantum electrodynamics is characterized by its Lorentz symmetry; however Maxwellian electrodynamics respects the larger conformal symmetry group. The use of conformal invariance has been helpful in the study of many intractable problems – can it be used to help approximate quantum processes with a non-linear theory? The group is defined by the multiplicative invariance of the line element under coordinate transformations [4].

\[
\eta^{\alpha\beta} dx^\alpha dx^\beta = \Omega^2(X,Y)\eta^{\alpha\beta} dx^\alpha dx^\beta.
\]

Conformal invariance reduces the Lagrangian density to,

\[
\mathcal{L}_{\text{conformal}} = \frac{g}{4}(Y/X).
\]

The function \( \Omega \) depends only on the combination \( Y/X \) but is otherwise arbitrary. Any non-linear extension of electrodynamics must reproduce the results of Maxwell’s theory in the low field limit to be physically realistic; i.e. we require:

\[
\lim_{x \to 0} \Omega = 1.
\]

However, the restriction of the functional dependence being \( Y/X \) means that the only function satisfying this limit is unity. Maxwellian theory has a privileged status as the only physical conformal theory of electrodynamics. Conformal transformations can rescale a weak field into a strong field. A field described by Maxwell at low intensity must also be described by Maxwell at high intensity for conformally invariant theories.

Energy-momentum

The energy-momentum tensor of waves interacting with a real medium has two conflicting descriptions in the literature – Minkowski and Abraham (see [5] for review). Non-linear theories cause the vacuum to behave analogously to a dielectric material, but are simpler to work with. Can the problem be resolved in this context? The energy-momentum tensor in non-linear electrodynamics is given by [6],

\[
T^{\mu\nu} = H^{\mu\nu} F^{\mu\nu} - \partial_\mu \partial_\nu \mathcal{L}.
\]

In the usual context, the energy-momentum is partitioned into wave and material parts. The analogous case is to separate the full electromagnetic field into a strong background and weak probe. \( F^{\mu\nu} = F^{\mu\nu} + f^{\mu\nu} \), such that \( H^{\mu\nu} = H^{\mu\nu} + h^{\mu\nu} + \kappa f^{\mu\nu} + O(f^2) \). Expanding the non-linear energy-momentum,

\[
T^{\mu\nu} = T^{\mu\nu}_W + T^{\mu\nu}_M + f^{\mu\nu} f_{\mu\nu} + \kappa f^{\mu\nu} f_{\mu\nu} + \mathcal{O}(f^3).
\]

The \( \mathcal{O}(f^3) \) contribution is naturally defined in terms of the Minkowski energy-momentum, which is not symmetric in general. However, the antisymmetric part of the final term in the expansion above exactly cancels the non-symmetric part of the Minkowski term, ensuring the total is symmetric. Appealing to Lorentz invariance of the theory also identifies Minkowski as the best description [7].

Non-linear generation of fields

Taking a background field \( F^{\mu\nu} \) for which the invariants \( X, Y \) vanish – such as a plane-wave then both,

\[
\partial_\mu F^{\mu\nu} = 0 \quad \text{and} \quad \partial_\mu H^{\mu\nu} = 0,
\]

are satisfied. The superposition of two plane-waves is also a solution of Maxwell’s equations. Defining two counter-propagating linearly polarized plane-waves,

\[
F^{\mu\nu}(\phi) = [(k^0 \epsilon^\mu - k^\mu \epsilon^0) E^\nu(\phi)], \quad \phi = k_x x,
\]

the non-linear interaction between these will generate a new field \( f^{\mu\nu} \), satisfying,

\[
\partial_\mu f^{\mu\nu} = -\partial_\nu H^{\mu\nu} = j^\nu.
\]

This spreads out with the initial fields, so the total outgoing field in the direction of the pulse with initial momentum \( k_1 \) will be,

\[
f^{\mu\nu}(\phi) = [(k^0 \epsilon^\mu - k^\mu \epsilon^0) E^\nu(\phi)] - \frac{1}{2} \delta^\nu_\rho \int d\phi E^\rho(\phi) E^\nu(\phi)
\]

with \( \beta \) a theory dependent constant. This essentially looks like the first term in a Taylor expansion of the initial field. The outgoing wave will be delayed by an amount proportional to the fluence of the \( k_1 \) direction wave.