ANALYTICAL MODEL FOR NON-UNIFORM CORROSION-INDUCED CONCRETE CRACKING

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ABSTRACT

Corrosion-induced concrete cracking is one of the major deterioration mechanisms for reinforced concrete structures. Evidently, the corrosion process is not uniform along the circumference of the reinforcement. To model the stress distribution in concrete and determine the initiation of concrete cracking, a realistic non-uniform corrosion model needs to be developed. In this paper, a time-dependent corrosion model, producing non-uniform expansion to concrete, is first established. The stresses in concrete are then formulated through the employment of complex functions. The time to initiation of concrete cracking is determined and related to a number of material, geometric and corrosion parameters. The derived analytical model is verified by comparing the results with those from experimental tests in literature. It can be concluded that this model is one of very few analytical models that can determine the stresses in concrete caused by non-uniform corrosion of reinforcement in concrete.

KEYWORDS: Durability-related properties; Fracture & fracture mechanics; Cracks & cracking.

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INTRODUCTION

Corrosion of reinforcement is a significant problem affecting the durability of reinforced concrete (RC) structures. Practical experience and experimental observations (Andrade et al., 1993, Li, 2003, Otsuki et al., 2000) suggest that corrosion affected RC structures deteriorate faster in terms of serviceability (e.g., cracking or deflection) than safety (e.g., strength). Consequently, corrosion can lead to premature deterioration of RC structures, causing concrete cracking, delaminating and de-bonding. For example, only 4% to 5% degree of corrosion (in terms of steel mass loss) can cause serviceability failure of RC structures as defined by corrosion-induced crack width (El Maaddawy et al., 2005). Moreover, the maintenance and repair costs for corrosion induced deterioration in RC bridges in the United States have been reported more than $5 billion per year (Koch et al., 2002).

Realising the significance of the problem, considerable research has been conducted during the last few decades. Early work mainly focused on uniform corrosion along the circumference of the reinforcing rebar in concrete (Liu and Weyers, 1998). This was probably because most experiments on producing corrosion in reinforced concrete employed impressed current technique; such an accelerated corrosion method controls the corrosion rate, hence the degree of corrosion, by adjusting the current and/or the time of interval applied to the reinforcement. As such, the corrosion generated by the impressed current method is in a uniform manner along the rebar. Meanwhile, almost all of the analytical and numerical studies assume a uniform corrosion expansion exerting between the reinforcement and its surrounding concrete. Under the corrosion expansion, concrete has been often modelled as a thick-wall cylinder (Bazant, 1979, Pantazopoulou and Papoulia, 2001, Li et al., 2006). Liu and Weyers (1998) proposed an analytical solution for the time to surface cracking by presuming concrete linear elastic and the
uniform corrosion development. This analytical model has soon been improved by a number of researchers, e.g., (Pantazopoulou and Papoulia, 2001, Li et al., 2006, Bhargava et al., 2006), who considered concrete as a quasi-brittle material, i.e., the fracture property of concrete being included in the models. These models can well address the corrosion-induced cracking behaviour, covering time to cracking initiation, time to surface cracking, surface crack width, etc. However, the limitation is that only uniform boundary condition can be applied to the formulation of the stress and strain in concrete.

More recently, modelling of corrosion-induced cracking of concrete has been focused on non-uniform corrosion progression at the interface between reinforcement and concrete. Due to the fact that chlorides, as well as moisture and oxygen, penetrates to the depth of the reinforcement at different rates on different sides of the concrete, it is very rare to have a uniform and general corrosion on the reinforcement. It has been reported (González et al., 1995) that the pitting caused localized deterioration is equivalent to about four to eight times that of the reinforcement under overall corrosion. To study the non-uniform corrosion caused structural deterioration, numerical approach, mainly finite element method (FEM), has played a dominating role (Jang and Oh, 2010, Pan and Lu, 2012, Zhao et al., 2011, Du et al., 2014). Jang and Oh (2010) considered a few of non-uniform distributions of corrosion products and simulated the stress states in concrete accordingly. Pan and Lu (2012) modelled the non-uniform corrosion caused crack propagation in concrete with FEM and the concrete as a heterogeneous material. Du et al. (2014) employed damage plasticity model to simulate the concrete cracking under non-uniform corrosion induced expansion. A number of parameters were investigated on their effects to the surface cracking. Zhao et al. (2011) proposed a Gaussian distribution for the non-uniform corrosion caused displacement and simulated the cracking behaviour of concrete by smeared crack model. Literature review suggests (1) the
distribution of the non-uniform corrosion caused expansion has seldom been based on experimental test results but more on assumptions and; (2) no model has been developed in relating concrete cracking to some basic parameters for the non-uniform corrosion, e.g., corrosion rate. As pointed out in (Zhao et al., 2011), this is probably because there is an absence of reliable data which can characterize the actual non-uniform formation and expansion of the corrosion products.

Yuan and Ji (2009) conducted the accelerated corrosion tests on reinforced concrete samples in an artificial environmental chamber and obtained a non-uniform corrosion distribution along the interface between reinforcement and concrete. In their findings, only a half of the reinforcement, facing concrete cover, was corroded and the corrosion caused expansion was in a semi-elliptical shape. Amongst limited experimental data, the non-uniform distribution derived from Yuan and Ji (2009) can provide the reasonable inputs for modelling the non-uniform corrosion caused concrete cracking. Moreover, although numerical approaches are more powerful in terms of solving a wider range of problems, analytical solutions are more accurate and convenient for the practical application. It would be, therefore, ideal to have an analytical model on non-uniform corrosion induced cracking of concrete. In the analytical model, some key material and corrosion parameters, e.g., corrosion rate, can be related to the structural behaviour, e.g., time to cracking. However, the non-uniform stress distribution, along the hoop direction of the concrete cylinder, requires complex functions to be used in the analytical formulation. In this case, concrete can only be treated as an elastic material, with cracking initiation being modelled. The initiation of cracking in concrete marks the start of the structural deterioration of reinforced concrete structures. After that the structure usually degrades faster and reaches to limit states quickly.
This paper attempts to develop an analytical model for non-uniform corrosion induced concrete cracking. A time-dependent non-uniform corrosion model is first derived, based on realistic experimental results from literature. A cracking model is then formulated by using the complex functions to account for the non-uniform stress distribution on concrete. A number of material, geometric and corrosion parameters are considered in the analytical model. The initiation of the concrete cracking is determined as a function of its service time.

RESEARCH SIGNIFICANCE

Although considerable research has been conducted on modelling corrosion-induced concrete cracking problems, very few models have been proposed based on non-uniform corrosion of reinforcement in concrete. Amongst these available models for non-uniform corrosion, most employed numerical approaches while almost none on an analytical manner. However, it has become a widely accepted fact that the corrosion of reinforced concrete is normally not uniform. For accurate prediction of cracking caused by corrosion, it is necessary to consider non-uniform corrosion expansion in the formulation of stress and strain in the concrete solid. It is therefore imperative to derive a rational model for corrosion-induced concrete cracking to achieve the cost effectiveness in the asset management of reinforced concrete structures. To the best knowledge of the authors, this paper represents the first attempt in the analytical formulation of time-dependent non-uniform corrosion induced concrete cracking.

NON-UNIFORM CORROSION MODEL

Concrete with an embedded bar subjected to an internal pressure at the interface between the bar and concrete can be modelled as a thick-wall cylinder (Bazant, 1979, Pantazopoulos and Papoulias, 2001, Tepfers, 1979). This is schematically shown in Figure 1(a), where $D$ is the
diameter of the bar; \( d_0 \) is the thickness of the annular layer of concrete pores at the interface between the bar and concrete; and \( C \) is the concrete cover. Usually \( d_0 \) is constant once concrete has hardened. The inner and outer radii of the cylinder are \( a = D/2 + d_0 \) and \( b = C + D/2 + d_0 \).

The corrosion products (mainly ferrous and ferric hydroxides, \( Fe(OH)_2 \) and \( Fe(OH)_3 \)) occupy a few times more space than the original steel. The corrosion products first fill in the annular pores in concrete around the reinforcing bar, with thickness \( d_0 \), but normally do not produce stresses in concrete. As corrosion propagates in concrete, a band of corrosion products forms, as shown in Figure 1(b). If the reinforcement corrosion is a uniform process along the circumference of the reinforcing bar, the band becomes a circular ring which causes uniform expansive pressure on the concrete cylinder (Li et al., 2006). However, this is usually not realistic as discussed earlier. It has been found that (Yuan and Ji, 2009), the front of corrosion products for the half of rebar facing concrete cover is in a semi-elliptical shape, while corrosion of the opposite half of rebar is negligibly small and can be neglected.

The total amount of corrosion products \( W_{rust}(t) \) can be assumed to distribute around the bar, occupying three parts as shown in Figure 1(c): the semi-elliptical band of corroded steel with maximum thickness \( d_{co-st} \), the porous circular band \( d_0 \) and the semi-elliptical rust band with maximum thickness \( d_m \). It should be noted that the semi-major axis for the semi-ellipse of corrosion front is \( D/2 + d_0 + d_m \). \( W_{rust}(t) \) can thus be expressed as follows:

\[
W_{rust}(t) = W_s + W_0 + W_m
\]  \hfill (1)
Where $W_s$ is the amount of rust replacing the corroded steel, $W_0$ is the amount of rust filling the porous band $d_0$ and $W_m$ is the amount of rust in the band $d_m$. $W_s$, $W_0$ and $W_m$ can be derived respectively as follows:

$$W_s = \alpha_{rust} W_{rust} \frac{\rho_{rust}}{\rho_{st}}$$  \hspace{1cm} (2)

$$W_0 = \frac{\pi \rho_{rust} (D + d_0) d_0}{2}$$  \hspace{1cm} (3)

$$W_m = \rho_{rust} \left[ \frac{\pi}{2} \left( \frac{D}{2} + d_0 \right) \left( \frac{D}{2} + d_0 + d_m \right) - \frac{\pi}{2} \left( \frac{D}{2} + d_0 \right)^2 \right] = \frac{\pi \rho_{rust}}{2} \left( \frac{D}{2} + d_0 \right) d_m$$  \hspace{1cm} (4)

$\alpha_{rust}$ is the molecular weight of steel divided by the molecular weight of corrosion products. It varies from 0.523 to 0.622 according to different types of corrosion products (Liu and Weyers, 1998). $\rho_{rust}$ is the density of corrosion products.

By substituting Equations (2-4) into Equations (1), it becomes:

$$\frac{2W_{rust}(t)}{\pi} \left( \frac{1}{\rho_{rust}} - \frac{\alpha_{rust}}{\rho_{st}} \right) = D d_0 + d_0^2 + \frac{D}{2} d_m + d_0 d_m$$  \hspace{1cm} (5)

By neglecting the second order of small quantities, i.e., $d_0 d_m$ and $d_0^2$, $d_m$ can be derived as follows:

$$d_m(t) = \frac{4W_{rust}}{\pi D} \left( \frac{1}{\rho_{rust}} - \frac{\alpha_{rust}}{\rho_{st}} \right) - 2d_0$$  \hspace{1cm} (6)

$d_m(t)$ in Equation (6) is the maximum corrosion-induced expansion along the interface to the concrete cylinder under which the stress will be initiated in the cylinder. $d_m(t)$ determines the...
shape of the semi-ellipse which is the boundary condition of the concrete cylinder in deriving stresses and strains in concrete.

In Equation (6), $W_{\text{rust}}(t)$ is related to the corrosion rate of the steel rebar and can be expressed as (Liu and Weyers, 1998):

\[
W_{\text{rust}}(t) = \sqrt{\frac{t}{0}} 0.105(1/\alpha_{\text{rust}})\pi D_{\text{cor}}(t)dt
\]

(7)

where $i_{\text{cor}}$ is the corrosion current density in $\mu A/cm^2$, which is widely used as a measure of corrosion rate.

The units of the parameters in Equations (1-7) need to keep consistent. For the clarification and also the convenience of readers, the units are specified in Table 1.

To determine the displacement boundary condition of the concrete cylinder, the function of the semi-ellipse of the corrosion front needs to be derived. It is known that, in rectangular coordinate system, the function for an ellipse can be expressed as follows:

\[
\frac{y^2}{A_L^2} + \frac{x^2}{A_S^2} = 1
\]

(8)

where $A_L$ is the half length of major axis and $A_S$ is the half length of the minor axis for an ellipse, as shown in Figure 2. The bottom half of the circular band $d_0$ is shaded for easier recognition of the three bands, e.g., $d_{\text{co-st}}$, $d_0$ and $d_m$. Based on the geometry in Figure 2, $A_L$ and $A_S$ can be obtained, i.e., $A_L = D/2 + d_0 + d_m$ and $A_S = D/2 + d_0$.

Transforming to a polar coordinate system, Equation (8) can be rewritten as follows,
\[ r = \frac{A_L A_s}{\sqrt{A_s^2 \sin^2 \theta + A_L^2 \cos^2 \theta}} \]  

(9)

Substituting \( A_L \) and \( A_s \) in the above equation,

\[ r = \frac{(D + 2d_0 + 2d_m)(D + 2d_0)}{\sqrt{(2D + 4d_0)^2 + 16d_m(D + 2d_0 + d_m)\cos^2 \theta}} \]  

(10)

The displacement boundary condition of the concrete cylinder \( \delta(\theta, t) \) can therefore be derived, with \( d_m \) substituted in Equation (6):

\[
\delta(\theta, t) = \frac{\left(D + 2d_0 + \frac{8W_{rust}(t)}{\pi D} \left(\frac{1}{\rho_{rust}} - \frac{\alpha_{rust}}{\rho_{st}}\right) - 4d_0\right)(D + 2d_0)}{\sqrt{(2D + 4d_0)^2 + 32 \left\{(\frac{1}{\rho_{rust}} - \frac{\alpha_{rust}}{\rho_{st}}) - d_0\right\} D + 2d_0 + \frac{4W_{rust}(t)}{\pi D} \left(\frac{1}{\rho_{rust}} - \frac{\alpha_{rust}}{\rho_{st}}\right) - 2d_0}\cos^2 \theta} - \frac{D}{2} - d_0
\]  

(11)

where \( 0 \leq \theta \leq \pi \).

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**CONCRETE CRACKING MODEL**

As discussed, the concrete is modelled as a thick wall cylinder. In polar coordinate system, the stress components in plane stress/strain elastic body can be expressed in terms of two arbitrary complex functions \( \gamma(z) \) and \( \psi(z) \) (Sadd, 2005) as follows:

\[ \sigma_r + \sigma_\theta = 2\left|\gamma'(z) + \gamma(z)\right| \]  

(12)

\[ \sigma_\theta - \sigma_r + 2i\tau_{r\theta} = 2\left|z\gamma'(z) + \psi(z)\right|e^{2i\theta} \]  

(13)

where \( \sigma_r \) is the radial stress component, \( \sigma_\theta \) is the hoop stress component and \( \tau_{r\theta} \) is the shear stress component.

Solving the Equations (12) and (13) the individual stress components can be derived as follows,
\[ \sigma_r = \text{Re}\left[2\gamma'(z) - \bar{z}\gamma^*(z)e^{2i\theta} - \psi'(z)e^{2i\theta}\right] \]  
(14)

\[ \sigma_\theta = \text{Re}\left[2\gamma'(z) + \bar{z}\gamma^*(z)e^{2i\theta} + \psi'(z)e^{2i\theta}\right] \]  
(15)

\[ \tau_{r\theta} = \text{Im}\left[\bar{z}\gamma'(z)e^{2i\theta} + \psi'(z)e^{2i\theta}\right] \]  
(16)

The solution to the complex functions \( \gamma(z) \) and \( \psi(z) \) relies on solving the boundary conditions of the hollow circular cylinder. There are two types of boundary conditions that can be formulated, i.e., stress boundary condition and displacement boundary condition. These conditions can be expressed as follows:

\[ \gamma(z) + z\gamma'(z) + \bar{\psi}(z) = iF(z) \quad \text{for stress boundary condition} \]  
(17)

\[ \kappa\gamma(z) - z\gamma'(z) - \bar{\psi}(z) = 2G\delta(z) \quad \text{for displacement boundary condition} \]  
(18)

where \( F(z) \) is the resultant force exerting on a boundary, \( G \) is the shear modulus, \( \delta \) is the displacement exerting on a boundary. \( \kappa = 3 - 4\nu \) for plane strain problem and \( \kappa = \frac{3 - \nu}{1 + \nu} \) for plane stress problem, while \( \nu \) is the Poisson’s ratio.

As discussed, the expansive mechanism is modelled as displacement boundary condition to the inner circle of the concrete cylinder. The inner boundary condition can be expressed as follows,

\[ \kappa\gamma(z_1) - z_1\gamma'(z_1) - \bar{\psi}(z_1) = 2G\delta(z_1, r), \quad z_1 = ae^{i\theta} \]  
(19)

The outer boundary is the surface of the concrete cover. Therefore, the stress condition can be formulated as follows,

\[ \gamma(z_2) + z_2\gamma'(z_2) + \bar{\psi}(z_2) = iF(z_2) = 0, \quad z_2 = be^{i\theta} \]  
(20)
According to Laurent’s Theorem, each complex potential can be expressed as a power series, reducing the stress and/or displacement boundary problems to sets of simultaneous linear equations in the coefficients of the two power series (MuSkhelishvili, 1953).

\[ \gamma(z) = \sum_{-\infty}^{\infty} \alpha_n z^n \]  

(21)

\[ \psi(z) = \sum_{-\infty}^{\infty} \beta_n z^n \]  

(22)

where \( a \leq |z| \leq b \). Substituting Equations (21) and (22) into the stress boundary condition, i.e., Equation (20), it becomes

\[ \sum_{-\infty}^{\infty} \alpha_n z_2^n + z_2 \sum_{-\infty}^{\infty} n \alpha_n z_2^{n-1} + \sum_{-\infty}^{\infty} \beta_n z_2^n = 0 \quad \text{for} \quad z_2 = be^{i\theta} \]  

(23)

For the sake of simplifying Equation (23), both sides of the equation are multiplied by \( e^{-im\theta} \) (\( m = 0 \) or integer), followed by integration with respect to \( \theta \) from 0 to \( 2\pi \). The boundary condition (Equation 23) then becomes:

\[ \sum_{-\infty}^{\infty} \alpha_n b^n \int_{0}^{2\pi} e^{in\theta} e^{-im\theta} d\theta + be^{i\theta} \sum_{-\infty}^{\infty} n \alpha_n b^{n-1} \int_{0}^{2\pi} e^{-(n-1)m\theta} e^{-im\theta} d\theta + \sum_{-\infty}^{\infty} \beta_n b^n \int_{0}^{2\pi} e^{-im\theta} e^{-im\theta} d\theta = 0 \]  

(24)

It can be mathematically proved that, \( \int_{0}^{2\pi} e^{in\theta} d\theta = 0 \) if \( n \) is an integer or \( 2\pi \) if \( n = 0 \). Therefore Equation (24) can be simplified as follows:

\[ b^{2m} \alpha_m + (2 - m)b^2 \alpha_{2-m} + \beta_{-m} = 0 \]  

(25)

Similarly, by substituting the power series, the inner displacement boundary condition, i.e., Equation (19), becomes,

\[ \kappa \sum_{-\infty}^{\infty} \alpha_n z_1^n - z_1 \sum_{-\infty}^{\infty} n \alpha_n z_1^{n-1} - \sum_{-\infty}^{\infty} \beta_n z_1^n = 2G \delta(\theta, t), \quad \text{for} \quad z_1 = ae^{i\theta} \]  

(26)
Substituting $z_i = ae^{i\theta}$, multiplying $e^{-im\theta}$ and integrating with respect to $\theta$ from 0 to $2\pi$,

Equation (26) can be simplified as follows:

$$
\kappa a^{2m}\alpha_m - (2-m)\beta_m + \beta_m = \frac{a^m G}{\pi} \int_0^{2\pi} \delta(\theta, t)e^{-im\theta} d\theta
$$

By combining Equation (25) and (27), $\alpha_m$ and $\beta_m$ can be derived.

$$
\alpha_m = \frac{(2-m)(a^2 - b^2)a^{-2m}G \int_0^{2\pi} \delta(\theta, t)e^{i(2m)\theta} d\theta + \frac{\kappa a}{\pi}a^{-2m}G \int_0^{2\pi} \delta(\theta, t)e^{-im\theta} d\theta}{\pi(ka^{2m} + b^{2m}) - m(2-m)(a^2 - b^2)}
$$

$$
\beta_m = \frac{\pi[ka^{2(m+1)} + b^{2(m+1)}] \int_0^{2\pi} \delta(\theta, t)e^{-im\theta} d\theta}{\pi(a^2 + b^2)}
$$

To solve $\alpha_m$ and $\beta_m$ with regards to the integrals in Equations (28) and (29), $\delta(\theta, t)$, i.e.,

Equation (11), needs to be expanded to Fourier series as follows (MuSkhelishvili, 1953):

$$
\delta(\theta, t) = \sum_{k=-\infty}^{\infty} A_k e^{ik\theta}
$$

where $A_k = \frac{1}{2\pi} \int_0^{2\pi} \delta(\theta, t)e^{-ik\theta} d\theta$.

It is difficult to derive the explicit integrated form of series coefficients $A_k$; however, those coefficients can be determined by numerical integration once all the other parameters are evaluated.

Substituting Equations (30) into Equations (28) and (29),

- 12 -
\[\alpha_m = \frac{(2 - m)(a^2 - b^2)a^{2-m}G\sum_{n=0}^{\infty} A_n \int_{0}^{2\pi} e^{i(2-k-m)\theta}d\theta + \left[ka^{2(2-m)} + b^{2(2-m)}\right]a^m G\sum_{n=0}^{\infty} A_k \int_{0}^{2\pi} e^{-i(k+m)\theta}d\theta}{\pi\left[ka^2 + b^2\right]\left[ka^{2(2-m)} + b^{2(2-m)}\right] - m(2 - m)(a^2 - b^2)^2}\]

(31)

\[\beta_m = \frac{\pi\left[ka^{-2(m+1)} + b^{-2(m+1)}\right]a^{-m} + a^{-(m+2)}G\sum_{n=0}^{\infty} A_k \int_{0}^{2\pi} e^{-i(k+m)\theta}d\theta}{\pi(a^2 + b^2)}\]

(32)

Since \(\int_{0}^{2\pi} e^{in\theta}d\theta = 0\) if \(n\) is an integer or \(2\pi\) if \(n = 0\), \(\alpha_m\) and \(\beta_m\) can be determined.

\[\alpha_m = \frac{2(2 - m)(a^2 - b^2)a^{2-m}G\sum_{n=0}^{\infty} A_n + 2\left[ka^{2(2-m)} + b^{2(2-m)}\right]a^m GA_m}{\left[ka^{2m} + b^{2m}\right]\left[ka^{2(2-m)} + b^{2(2-m)}\right] - m(2 - m)(a^2 - b^2)^2}\]

\[\beta_m = \frac{\left[ka^{-2(m+1)} + b^{-2(m+1)}\right]a^{-m} + 2a^{-(m+2)}GA_m}{a^2 + b^2}\]

(33)

(34)

where \(m\) is an integer or 0. The Fourier coefficients \(A_k\) can be numerically determined in MatLab and therefore \(\alpha_m\) and \(\beta_m\) are solved. By substituting \(\alpha_m\) and \(\beta_m\) into Equations (21) and (22), the stress components, i.e., \(\sigma_r\), \(\sigma_\theta\) and \(\tau_{r\theta}\), can be calculated from Equations (14) – (16).

WORKED EXAMPLE

To demonstrate the application of the derived model, a numerical example is undertaken on a reinforced concrete beam originally investigated in (Li, 2003). The corrosion was achieved via saltwater spray in a customer designed environmental chamber. The values of basic variables of the structure and corrosion are shown in Table 2.
With these values of the basic variables, first of all, the corrosion caused expansion (displacement) can be obtained. $d_m(t)$ in Equation (6), defining the shape of the inner displacement boundary condition of the concrete cylinder, is illustrated as a function of service time (10 years), shown in Figure 3. Such a development of corrosion expansion is based on the assumption that the corrosion rate $i_{corr}$ is time-dependent and equal to $0.3686 \ln(t) + 1.1305 \mu A/cm^2$ (Li, 2003). The inset in Figure 3 represents an enlarged picture showing the initial progression of the corrosion products. It can be seen that the maximum displacement $d_m(t)$ starts to increase from zero at around 0.2 year, which is then followed by gradual increase in the subsequent lifetime. Once $d_m(t)$ is obtained, the whole inner boundary condition of the concrete cylinder, represented by Equation (11), can be determined.

Under the non-uniform expansion caused by reinforcement corrosion, the stresses can be calculated from Equations (14) – (16). Figures 4 and 5 show the hoop and radial stresses, respectively, at the inner boundary of the concrete cylinder at 0.23 year. As illustrated in Figure 1 – c, the point of zero degree is located at the right middle of the inner boundary, with $[0, -\pi/2]$ represents the top right quarter and $[0, \pi/2]$ represents the bottom right quarter, of the concrete cylinder. The whole cylinder is symmetric against the y-axis, as shown in Figure 2. It can be seen that the hoop stress changes along the inner boundary, most of which is under tension. It has been found the highest hoop stress occurs at the location close to 15°. This is the location where the crack is initiated. Although most of the concrete is in tension, a small part of the concrete cylinder, from 80° to 90°, is under mild compression in the hoop direction. It is worth to mention that the maximum displacement $d_m(t)$ of the elliptical shape of the boundary condition is added at the location of 90°. The combined tension and compression distribution of hoop stress is quite different than previous findings by assuming uniform corrosion expansion.
where the hoop stresses of the concrete are all in tension, e.g., in (Li and Yang, 2011). Under the non-uniform expansion, the point at 90° location appears to have the largest radial and compressive stress. At the region around 15° the radial stress is in tension. It is very interesting to find, under the non-uniform corrosion expansion, the 90° location is in compression in both hoop and radial directions, whilst the 15° location is in tension in both directions. It should be noted that the positive hoop stress in Figure 4 represents tensile stress while the positive radial stress in Figure 5 represents compressive stress.

VALIDATION OF THE DEVELOPED MODEL

The derived model is verified by comparing the time to cracking initiation of the model and the experimental results from literature. According to the literature searched, almost all the test data regarding the time to cracking initiation are based on uniform corrosion development by electric current method for accelerated corrosion. For the limited experimental research on non-uniform corrosion by utilizing artificial environmental chamber with salt spray function, no data on time to cracking initiation was provided. In light of comparing the derived model with the test results, a special case of uniform corrosion was solved based on the developed model. The stresses were then computed by the derived analytical equations and the time to cracking initiation was determined, according to given tensile strength of the concrete. The values for all inputs were made the same between the analytical model and the experiment. The corrosion rate \( i_{corr} \) is time-dependent in this model; but for the purpose of comparison, \( i_{corr} = 100 \mu A/cm^2 \) was used as was applied in the test (Mullard and Stewart, 2011). The comparison to the experimental results is shown in Table 3. The results from the analytical model and the experimental tests are in reasonable agreement. Further, it is interesting to find that all the modelled times to cracking initiation are larger than the experimental ones. This is
probably because the assumption of the ITZ in the model, i.e., it is totally stress-free when the corrosion products are filling the band of ITZ. In fact, however, the filling of the corrosion products in the porous ITZ can cause pressure. As a result, such an assumption of the ITZ will underestimate the pressure induced by corrosion. The effect of the thickness $d_0$ of the ITZ will be elaborated in the following section and illustrated in Figure 8.

**ANALYSIS AND DISCUSSION**

Corrosion rate, normally expressed as corrosion current density $i_{corr}$, has been considered one of the key factors affecting the durability of reinforced concrete structures. A number of researchers have been working on developing realistic models for the corrosion rate; in the worked example, a time-dependent corrosion rate was employed. To investigate the effects of the corrosion rate on the corrosion caused expansion, i.e., $d_m(t)$, 4 constants corrosion rates are used, i.e., $i_{corr} = 0.5 \mu A/cm^2$, $i_{corr} = 1.0 \mu A/cm^2$, $i_{corr} = 5 \mu A/cm^2$ and $i_{corr} = 10 \mu A/cm^2$, as presented in Figure 6(a). These corrosion rates are believed to have covered a wide range of actual corrosion states of reinforced concrete structures. It can be seen that, as expected, the increase of the corrosion rate will cause the increase of expansive displacement to the concrete. The long-term effect is significant; for example, as the corrosion rate grows up to 10 times, the 10-year maximum displacement $d_m(t)$ could be increased to about 3.5 times. Such an increase magnitude in the displacement boundary condition is crucial to the stress development in the concrete cylinder. Nevertheless, the short-term effect is more sensitive, as shown in Figure 6(b). The corrosion rate $i_{corr} = 10 \mu A/cm^2$ causes immediate form and progression of the rust band at 0.01 year, whilst the pushing over of the rust band to the concrete cylinder is delayed to 0.19 year for the corrosion rate $i_{corr} = 0.5 \mu A/cm^2$. Further, the initial slope of the development
of $d_m(t)$ is considerably steeper for higher corrosion rates. This means the initiation of cracking might be very sensitive to the corrosion rate.

Figure 7 shows the development of $d_m(t)$ as the diameter of the reinforcing rebar changes. When the diameter of the rebar increases from 12mm to 20mm, the time to the initial progression of $d_m(t)$ is delayed slightly, i.e., 0.06 year. The long-term development of $d_m(t)$ against the service time is also reduced as the diameter of rebar increases. This proves that, for the same corrosion rate, larger size of rebar favours the corrosion caused expansion and hence delays the cracking initiation of the concrete.

Figure 8 demonstrates the effects of the ITZ thickness on the accumulation of the corrosion products. Three reasonable values of thickness are selected for the calculation of $d_m(t)$. As expected, the increase of the ITZ thickness can delay the occurrence of the expansive displacement. This is mainly because in this paper the porous ITZ is assumed to fully accommodate the corrosion products; as a result, the process of the accumulation of corrosion products in ITZ, does not cause any stress or displacement. In Figure 8, it has been found that the effect of ITZ thickness on the initial stage of $d_m(t)$ progression is quite considerable, with the delay of roughly 83% from 0.18 year to 0.33 year for $d_0 = 12e-6m$ and $d_0 = 20e-6m$, respectively. However, the long-term effects on $d_m(t)$ seems not very significant.

Figure 4 has shown the maximum tensile stress occurs at the location of around 15°, as a result of the non-uniform corrosion expansion modelled. It is therefore worth to plot the hoop stress
development history of this point. Figure 9 presents the hoop stress development as a function of service time for the point of 15°. The hoop stress for this point keeps constantly zero until 0.19 year which is followed by a very quick increase up to about 6MPa at 0.23 year. Most concrete will be cracked at such a level of tensile stress. The initial period of 0.19 year mainly represents the time for corrosion products propagating into the porous ITZ. Moreover, the initial 0.23 year represents for the initiation of cracking which marks the start of structural degradation.

Figure 10 shows the radial stress development for the point of 90° under the derived non-uniform corrosion model. This point is where the maximum displacement \( d_m(t) \) is applied and the maximum compressive stress occurs. Although the main failure type of this problem is fracture which is caused by tension, Figure 10 could be useful as to indicating the extent of compression at the time of cracking initiation.

By knowing the tensile strength of concrete \( f_t \), the time to cracking initiation can be obtained. This is based on the failure criterion of the tensile stress reaches the tensile strength of concrete. As specified in Table 2, \( f_t = 5.725 \text{MPa} \) is used in this study; hence the effect of corrosion rate on the time to initiation of cracking can be determined, as shown in Figure 11. Four corrosion rates are considered and fitted curve (\( R^2=1 \)) was produced according to these four points. It can be seen that for minor corrosion extent, e.g., \( i_{corr} \) is between 0.5\( \mu A/cm^2 \) and about 3.0\( \mu A/cm^2 \), the time to cracking initiation is very sensitive to the corrosion rate; however, when \( i_{corr} \) is larger than 3.0\( \mu A/cm^2 \), the change in time to initiation of cracking is very small.
CONCLUSIONS

In the paper, the stresses in concrete, subjected to non-uniform corrosion of the reinforcement, have been formulated. The non-uniform stress distribution was analytically solved by using the complex functions. The displacement boundary condition applied to the concrete cylinder was derived as a function of the time-dependent corrosion rate. Further, the time to initiation of concrete cracking has been determined based on the model developed, given values of the key material and corrosion parameters. It has been demonstrated that the developed analytical model can simulate the stresses in concrete caused by non-uniform corrosion and predict the time to cracking initiation. The model has also been partially verified due to the limitation of available test data on non-uniform corrosion directly. The developed model can be practically used by industrial engineers and asset managers for their decision-making in regards to corrosion induced concrete cracking problems.

ACKNOWLEDGEMENT

Financial support from European Commission via the Marie Curie IRSES project GREAT under FP7-PEOPLE-2013-IRSES-612665, Scottish Funding Council GRPe for early career researcher exchanges and Australian Research Council under DP140101547, LP150100413 and DP170102211 is gratefully acknowledged.
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<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unit</th>
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<tr>
<td>( D/d_0 )</td>
<td>mm</td>
</tr>
<tr>
<td>( \rho_{\text{rust}}/\rho_{\text{st}} )</td>
<td>( \text{kg/m}^3 )</td>
</tr>
<tr>
<td>( \alpha_{\text{rust}} )</td>
<td>n.a.</td>
</tr>
<tr>
<td>( W_{\text{rust}}/W_s/W_0/W_m )</td>
<td>( \text{mg/mm} )</td>
</tr>
<tr>
<td>( d_m(t) )</td>
<td>m</td>
</tr>
<tr>
<td>( i_{\text{corr}} )</td>
<td>( \mu \text{A/cm}^2 )</td>
</tr>
<tr>
<td>( t )</td>
<td>year</td>
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</table>
Table 2  Values of basic variables used in cracking computation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Values</th>
<th>Sources</th>
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<tr>
<td>$C$</td>
<td>31 mm</td>
<td>Li (2003)</td>
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<tr>
<td>$D$</td>
<td>12 mm</td>
<td>Li (2003)</td>
</tr>
<tr>
<td>$d_0$</td>
<td>12.5 μm</td>
<td>Liu and Weyers (1998)</td>
</tr>
<tr>
<td>$E_{ef}$</td>
<td>18.82 GPa</td>
<td>Li (2003)</td>
</tr>
<tr>
<td>$f_t$</td>
<td>5.725 MPa</td>
<td>Li (2003)</td>
</tr>
<tr>
<td>$i_{corr}$</td>
<td>$0.3686ln(t)+1.1305 \mu A/cm^2$</td>
<td>Li (2003)</td>
</tr>
<tr>
<td>$a_{rust}$</td>
<td>0.57</td>
<td>Liu and Weyers (1998)</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.18</td>
<td>Liu and Weyers (1998)</td>
</tr>
<tr>
<td>$G$</td>
<td>$E/[2(1+\nu)]$</td>
<td>Timoshenko and Goodier (1970)</td>
</tr>
<tr>
<td>$\rho_{rust}$</td>
<td>3600 kg/m$^3$</td>
<td>Liu and Weyers (1998)</td>
</tr>
<tr>
<td>$\rho_st$</td>
<td>7850 kg/m$^3$</td>
<td>Liu and Weyers (1998)</td>
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</table>
Table 3  Comparison of time to cracking initiation

<table>
<thead>
<tr>
<th>Specimen</th>
<th>D (mm)</th>
<th>C (mm)</th>
<th>$f_t$ (MPa)</th>
<th>$E_{ef}$ (GPa)</th>
<th>Time to cracking initiation from experiments (Mullard and Stewart, 2011) (hours)</th>
<th>Time to cracking initiation from the model (hours)</th>
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<tbody>
<tr>
<td>1</td>
<td>16</td>
<td>50</td>
<td>2.40</td>
<td>18.82</td>
<td>35</td>
<td>43</td>
</tr>
<tr>
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<td>50</td>
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<td>59</td>
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<td>18.82</td>
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<td>49</td>
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<td>4</td>
<td>27</td>
<td>36</td>
<td>3.79</td>
<td>18.82</td>
<td>43</td>
<td>68</td>
</tr>
</tbody>
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