A Theoretical Study of Two-Period Relaxations for Lot-Sizing Problems with Big-Bucket Capacities

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Abstract

In this paper, we study two-period subproblems proposed by [1] for lot-sizing problems with big-bucket capacities and nonzero setup times, complementing our previous work [3] investigating the special case of zero setup times. In particular, we study the polyhedral structure of the mixed integer sets related to various two-period relaxations. We derive several families of valid inequalities and investigate their facet-defining conditions. We also discuss the separation problems associated with these valid inequalities.

1 Introduction

In this study, we investigate multi-item production planning problems with big bucket capacities, i.e., each resource is shared by multiple items, which can be produced in a specific time period. These real-world problems are very interesting, as they remain challenging to solve to optimality and also to achieve strong bounds. The uncapacitated and single-item relaxations of the problem have been previously studied by [7]. The work of [6] introduced and studied the single-period relaxation with “preceding inventory”, where a number of cover and reverse cover inequalities are defined for this relaxation. Finally, we also note the relevant study of [5], who studied a single-period relaxation and compared with other relaxations.

We present a formulation for this problem following the notation of [2]. Let $NT$, $NI$ and $NK$ indicate the number of periods, items, and machine types, respectively. We represent the production, setup, and inventory variables for item $i$ in period $t$ by $x^t_i$, $y^t_i$, and $s^t_i$, respectively. We note that our model can be generalized to involve multiple levels as in [1], however, we omit this for the sake of simplicity.
\[
\begin{align*}
\min & \sum_{t=1}^{NT} \sum_{i=1}^{NI} f_i^t y_i^t + \sum_{t=1}^{NT} \sum_{i=1}^{NI} h_i^t s_i^t \\
\text{s.t.} & \ x_i^t + s_{i-1}^t - s_i^t = d_i^t & t \in \{1, \ldots, NT\}, i \in \{1, \ldots, NI\} \\
& \sum_{i=1}^{NI} (a_i^t \bar{x}_i^t + ST_k^i y_i^t) \leq C_k^t & t \in \{1, \ldots, NT\}, k \in \{1, \ldots, NK\} \\
& x_i^t \leq M_i^t y_i^t & t \in \{1, \ldots, NT\}, i \in \{1, \ldots, NI\} \\
& y \in \{0, 1\}^{NT \times NI}, x, s \geq 0
\end{align*}
\]

The objective function (1) minimizes total cost, where \(f_i^t\) and \(h_i^t\) indicate the setup and inventory cost coefficients, respectively. The flow balance constraints (2) ensure that the demand for each item \(i\) in period \(t\), denoted by \(d_i^t\), is satisfied. The big bucket capacity constraints (3) ensure that the capacity \(C_k^t\) of machine \(k\) is not exceeded in time period \(t\), where \(a_i^t\) and \(ST_k^i\) represent the per unit production time and setup time for item \(i\), respectively. The constraints (4) guarantee that the setup variable is equal to 1 if production occurs, where \(M_i^t\) represents the maximum number of item \(i\) that can be produced in period \(t\), based on the minimum of remaining cumulative demand and capacity available. Finally, the integrality and non-negativity constraints are given by (5).

## 2 Two-Period Relaxation

Let \(I = \{1, \ldots, NI\}\). We present the feasible region of a two-period, single-machine relaxation of the multi-item production planning problem, denoted by \(X^{2PL}\) (see [1] for details).

\[
\begin{align*}
& x_{i'}^t \leq \bar{M}_i^t y_{i'}^t & i \in I, t' = 1, 2 \\
& x_{i'}^t \leq \bar{d}_{i'}^t y_{i'}^t + s^i & i \in I, t' = 1, 2 \\
& x_1^t + x_2^t \leq \bar{d}_1^t y_1^t + \bar{d}_2^t y_2^t + s^i & i \in I \\
& x_1^t + x_2^t \leq \bar{d}_1^t + s^i & i \in I \\
& \sum_{i \in I} (a_i^t x_{i'}^t + ST_k^i y_{i'}^t) \leq \bar{C}_{i'}^t & t' = 1, 2 \\
& x, s \geq 0, y \in \{0, 1\}^{2 \times NI}
\end{align*}
\]

Since we consider a single machine, we dropped the \(k\) index from this formulation, however, all parameters are defined in the same lines as before. The obvious choice
for the horizon would be \( t+1 \), in which case the definition of the parameter \( \tilde{M}_t \) is the same as of the basic definition of \( M_{t+t'-1} \), for all \( i \) and \( t' = 1, 2 \). Similarly, capacity parameter \( \tilde{C}_t \) is the same as \( C_{t+t'-1} \), for all \( t' = 1, 2 \). Cumulative demand parameter \( \tilde{d}_t \) represents simply \( d_{t+t'-1,t+1} \), for all \( i \) and \( t' = 1, 2 \), i.e., \( \tilde{d}_1 = d_{1,2} \) and \( \tilde{d}_2 = d_{2} \). We note the following polyhedral result for \( X^{2PL} \) from [1].

**Proposition 2.1** Assume that \( \tilde{M}_i > 0, \forall t \in \{1, \ldots, NT\}, \forall i \in \{1, \ldots, NI\} \) and \( ST^t < \tilde{C}_t, \forall t \in \{1, \ldots, NT\}, \forall i \in \{1, \ldots, NI\} \). Then \( \text{conv}(X^{2PL}) \) is full-dimensional.

For the sake of simplicity, we will drop subscript \( t \) and symbol \( \sim \) in the following notations. In this paper, we investigate the case of \( a^t = 1, \forall i \in \{1, \ldots, NI\} \) with nonzero setups. We establish two relaxations of \( X^{2PL} \) and study their polyhedral structures. For a given \( t \), we define the first relaxation of \( X^{2PL} \), denoted by \( LR1 \), as set of \( (x, y) \in \mathbb{R}^{NI} \times \mathbb{Z}^{NI} \) satisfying

\[
\begin{align*}
  x^i &\leq M^i y^i, \ i \in I \\
  \sum_{i=1}^{NI} (x^i + ST^i y^i) &\leq C \\
  x^i &\geq 0, y^i \in \{0, 1\}, \ i \in I
\end{align*}
\]

Next, we present a result from the literature [4] concerning this relaxation.

**Definition 2.1** Let \( S_1 \subseteq I \) and \( S_2 \subseteq I \) such that \( S_1 \cap S_2 = \emptyset \). We say that \((S_1, S_2)\) is a generalized cover of \( I \) if \( \sum_{i \in S_1} (M^i + ST^i) + \sum_{i \in S_2} ST^i - C = \delta > 0 \).

**Proposition 2.2** (see [4]) Let \((S_1, S_2)\) be a generalized cover of \( I \), and let \( L_1 \subseteq I \setminus (S_1 \cup S_2) \) and \( L_2 \subseteq I \setminus (S_1 \cup S_2) \) such that \( L_1 \cap L_2 = \emptyset \). Then,

\[
\begin{align*}
  \sum_{i \in S_1 \cup L_1} x^i + \sum_{i \in S_2 \cup L_2} ST^i y^i - \sum_{i \in S_1} (M^i + ST^i - \delta)^+ y^i - \sum_{i \in S_2} (ST^i - \delta)^+ y^i \\
  - \sum_{i \in L_1} (\bar{q}^i - \delta) y^i - \sum_{i \in L_2} (ST^i - \delta)^- y^i &\leq C - \sum_{i \in S_1} (M^i + ST^i - \delta)^+ - \sum_{i \in S_2} (ST^i - \delta)^+ \leq C - \sum_{i \in S_1} (M^i + ST^i - \delta)^+ - \sum_{i \in S_2} (ST^i - \delta)^+
\end{align*}
\]

is valid for \( LR1 \), where \( A \geq \max(\max_{i \in S_1} (M^i + ST^i), \max_{i \in S_2} ST^i, \delta), \bar{q}^i = \max(A, M^i + ST^i), \) and \( \overline{ST}^i = \max(A, ST^i) \).

For a given \( t \), second relaxation of \( X^{2PL} \), denoted by \( LR2 \), can be defined as the set of \( (x, y, s) \in \mathbb{R}^{NI} \times \mathbb{Z}^{NI} \times \mathbb{R}^{NI} \) satisfying

\[
\begin{align*}
  x^i &\leq M^i y^i, \ i \in I \\
  x^i &\leq d^i y^i + s^i, \ i \in I \\
  \sum_{i=1}^{NI} (x^i + ST^i y^i) &\leq C \\
  x^i &\geq 0, y^i \in \{0, 1\}, \ s^i \geq 0, \ i \in I
\end{align*}
\]
In this talk, we will present the trivial facet-defining inequalities for \( LR^2 \), and then derive several classes of valid inequalities such as cover and partition inequalities. We will also present item- and period-extended versions of some of these families of inequalities, and we will establish facet-defining conditions for all families of inequalities. We will also discuss the separation problems associated with these valid inequalities.

References


